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Bacon's Strategy to Win the Infinite Hat Game: A Falsidical Paradox

Professor Rayo presented a paradoxical solution to the "infinite hat game" ("Hard Hat Problem"). It is similar to Example 1 in a paper by Bacon [1]. In this essay, I argue that the paradox is falsidical. The fallacy at the root of the paradox teaches us about the nature and limitations of the Axiom of Choice.

A countably infinite number of persons, P_k ($k \in \mathbb{N}$) are lined up in single file with P_1 at the back. Each person is assigned either a red or blue hat based on the flip of a fair coin. He can see all of the hats of higher-numbered people but cannot see his own hat or the hats of lower-numbered people. Everyone must simultaneously guess his color. If a person guesses correctly, he lives. If not, he is shot. Although no coordination is allowed at the time of guessing ("Game Day"), everyone can meet the night before and agree on a strategy ("Pre-Game Meeting"). It is uncertain that any pre-game strategy can improve on a naïve strategy in which each person randomly guesses a color and has a 50% likelihood of being shot. The expected number of deaths would be infinite.

Professor Rayo proposes a strategy to supposedly guarantee that only a finite number of people get shot. We represent any arbitrary assignment of hat colors as an omega sequence, s , of zeros and ones. A zero in the k th position of s means that P_k has a red hat and a one means that P_k has a blue hat. Ω is the set of all possible omega-sequences of zeros and ones. We define the "reference sequence," $s_R \in \Omega$, as the actual sequence of hat colors that is assigned by the coin flips on Game Day.

We partition Ω into non-overlapping "cells" or "equivalence classes." A cell is defined such that sequences s and s' in Ω are members of the same cell if and only if there are at most finitely many numbers k such that the s and s' differ in the k th position. The number of possible cells has the same cardinality as $\mathbb{R}$ and one can denote a cell with a real-valued index u (i.e., C_u where $u \in \mathbb{R}$).

Suppose that everyone can agree on a representative "guessing sequence", s_G , for each possible cell at the Pre-Game Meeting. The Axiom of Choice is invoked to argue that agreement on a representative s_G is possible. (Equivalently, they agree on a function f that maps any cell, C_u , to a guessing sequence, s_G , such that $s_G = f(C_u)$.)

In addition, suppose that everyone agrees to do the following on Game Day:

- (1) Person k observes the hat colors for all P_m where $m > k$. By doing this, he observes enough of the reference sequence, s_R , to determine the unique cell that corresponds to it.
- (2) He recalls the representative guessing sequence of that cell from the Pre-Game Meeting.

(3) He assumes that the guessing sequence, s_G , matches the reference sequence. Thus, his guess will be the value of the k th position s_G .

If everyone implements this strategy, it appears that only a finite number of people will be shot. This follows from the fact that the guessing sequence is in the same cell as the reference sequence and therefore the two sequences differ in only a finite number of positions. This is a paradox because each person only observes the colors of other hats, which are events that are independent of the color of his own hat, and it is difficult to understand how he gains information that improves the odds of guessing correctly to greater than 50%.

What went wrong? The problem is the use of the Axiom of Choice to argue that the people can agree on a representative guessing sequence for each cell. Here is one version of the axiom:

“AXIOM OF CHOICE: For every family F of nonempty sets, there is a function f such that $f(S) \in S$ for each set S in the family F . (We say that f is a choice function on F .)” [2]

In this problem, the family of nonempty sets is the countably infinite number of sets, $S_k = \{0, 1\}$, and $f(S_k)$ is choice of hat color for position k of the guessing sequence.

The problem is that the axiom refers to the existence of a choice function and says nothing about whether the function (or corresponding choice set) is unique and publicly knowable. (An axiom that stated that a unique and publicly knowable choice function and choice set must exist would be a “Super Duper” version of the Axiom of Choice.) Another way to phrase this is that the only way to guarantee that a function is unique and publicly knowable is to construct it, and the explicit purpose of the Axiom of Choice is to allow for non-constructive proofs.

In order to see the absurdity of believing that the Axiom of Choice implies that all persons can agree on a unique, publicly knowable representative for each cell, consider the minutes of the Pre-Game Meeting for cell C_u .

“CHAIR: Now we get to cell C_u . The Axiom of Choice guarantees that a guessing sequence exists for this cell, so just pick that sequence. Any questions?

AUDIENCE MEMBER: There are a countably infinite number of guessing sequences that match that cell. How can we know which one to pick without a construction method?

CHAIR: The Axiom of Choice is a magic wand and gives us the omniscient power to coordinate perfectly without constructing anything.”

If one accepts that the Pre-Game Meeting must specify a construction method in order to guarantee a unique, publicly knowable representative for a cell, one can then show that no such method exists. The argument proceeds by reductio. Suppose such a method exists. There are three possible ways to construct the value of the k th position of s_G :

(1) It can be pre-specified by creating an entry in a “dictionary” for cell C_u . The j th entry in the dictionary is an ordered pair (i_j, h_j) where i_j is an index (position number) and $h_j \in \{0,1\}$ is a hat color. If either $(k, 0)$ or $(k, 1)$ is in the dictionary, the value of position k will be 0 or 1, respectively.

(2) It can be determined by randomization (coin flip with bias p). (This is a generalization of (1) in which the dictionary maps indices to bias values instead of specific hat colors. The basic argument is unchanged.)

(3) It can be set equal to the value of position k of the reference sequence (i.e., observation of the actual hat color of P_k). This is the default method if index k is not found in the dictionary for the cell.

Now consider:

Lemma 1. The dictionary for any cell must have finite length.

Sketch of proof:

Step 1. Let s_R be a reference sequence, $C(s_R)$ be its cell, and $s_G(C(s_R))$ be the guessing sequence that is constructed from the cell. Now consider the value of position k of s_G . If this value is obtained from a dictionary, it cannot be guaranteed to match the value of position k of s_R . The reason is that if the dictionary specifies 0 (red), there is a 50% chance that the randomly chosen hat color will be 1 (blue) (and vice versa).

Step 2. Suppose that the dictionary has infinite length. It follows that there are an infinite number of positions at which s_G might not match s_R . (The expected number of discrepancies between the two sequences would be infinite). But this would mean that the reference and guessing sequences are not in the same cell.

Lemma 2. Let s_R be a reference sequence and $C(s_R)$ be its cell. There exists a finite number L such that the value for all positions $k > L$ of the guessing sequence must be obtained by direct observation of the corresponding position of the reference sequence.

Sketch of proof.

The length of the dictionary for $C(s_R)$ must be finite by Lemma 1. Define L as the maximum index i_j such that either $(i_j, 0)$ or $(i_j, 1)$ is in the dictionary. Then, if $k > L$, k will not be in the dictionary and its value must be generated by the default method of direct observation of position k of s_R .

Lemma 3. Let s_R be a reference sequence, $C(s_R)$ be its corresponding cell, and $L(C(s_R))$ be the finite number defined in Lemma 2. Now fix a value of k such that $k > L$ and consider the attempt of person P_k to implement the construction method for s_G agreed upon at the Pre-Game

Meeting. By Lemmas 1 and 2, he cannot use the dictionary to specify pre-determined values for the positions between $L+1$ and k , inclusive. But he cannot observe the values of s_R for these positions because he cannot see the hat colors of himself or people behind him!

This completes the outline of a proof that no method for constructing a guessing sequence can be specified at a Pre-Game Meeting and implemented fully by every person. Moreover, if $k > L$ as defined above, person P_k cannot construct the value of the guessing sequence at the only position that matters to him, namely position k . Thus, it is not clear that his guess of hat color is even defined, let alone that it matches the pre-game consensus that is meant to ensure that only a finite number of people get shot.

References:

[1] Bacon, Andrew, "A paradox for supertask decision makers," *Philosophical Studies*, 2011; 153(2), p. 307.

[2] Jech, Thomas J., *The Axiom of Choice*, Mineola, NY: Dover Publications, Inc., 2008, p. 1.